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AN EFFICIENT ALGORITHM TO RECOGNIZE LOCALLY EQUIVALENT GRAPHS

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To locally complement a simple graph F at one of its vertices v is to replace the subgraph induced by F on $n(v)=\{w : vw$ is an edge of $F\}$ by the complementary subgraph. Graphs related by a sequence of local complementations are said to be locally equivalent. We associate a system of equations with unknowns in $GF(2)$ to any pair of graphs $\{F,F'\}$, so that F is locally equivalent to F' if and only if the system has a solution. The equations are either linear and homogenous or bilinear, and we find a solution, if any, in polynomial time.

1. Local equivalence

Let F be a simple graph. The *neighborhood* of a vertex v of F is $n(v)=\{w : vw$ is an edge of $F\}$. To *locally complement* F at v is to replace the subgraph induced by F on $n(v)$ by the complementary subgraph. We denote by $F*v$ the local complement of F at v. Clearly

$$(F*v)*v=F.$$

For a word $v_1v_2\dots v_q$ with letters in V we define

$$F*(v_1v_2\dots v_q)=(((F*v_1)*v_2)\dots)*v_q$$

and we say that $F'=F*(v_1v_2\dots v_q)$ is *locally equivalent* to F. This is actually an equivalence relation because the above equality implies $F=F'*(v_q\dots v_2v_1)$. We notice that locally equivalent graphs are defined over the same vertex-set.

Local complementations are natural operations in the following situation. Let m be a word on a set of letters V and suppose that each letter precisely occurs twice in m. An *alternance* of m is a non-ordered pair xy of letters such that we alternatively meet $\dots x\dots y\dots x\dots y\dots$ or $\dots y\dots x\dots y\dots x\dots$ when reading m. The simple graph on the vertex-set V whose edges are the alternances of m is denoted by $A(m)$ and called the *alternance graph* of m. For example if $m=0410213243$ then $A(m)$ has edges 01, 12, 23, 34, 40. Consider some $v\in V$, the decomposition $m=PvQvR$ where P, Q, R are subwords of m, and $m*v=Pv\tilde{Q}vR$ where $\tilde{Q}$ is the mirror-image of Q. With the preceding example we have $m*1=0412013243$. It is easy to verify that $A(m*v)=A(m)*v$. This interpretation was introduced by A. Kotzig [6], and we give further details in [2]. Not every simple graph is an alternance graph. For example the 5-wheel is not.

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2. Isotropic systems

This section recalls background properties proved in [3], except (2.5) proved in [4].

For any finite set V, we consider $\mathcal{P}(V)$, the power-set of V, with its canonical structure of vector-space over $GF(2)$. Thus for $X, Y \subseteq V$, $X + Y$ is the symmetric difference of X and Y. The *neighborhood function* of a simple graph F over the vertex-set V is the linear function $n : \mathcal{P}(V) \to \mathcal{P}(V)$ such that $n(v) = \{w : vw$ is an edge of $F\}$, $v \in V$.

Let K denote a 2-dimensional vector space over $GF(2)$, provided with the bilinear form given by $\langle x, y\rangle = 1$ if and only if $0 \neq x \neq y \neq 0$. For any finite set V we consider that the $2|V|$-dimensional vector space K^V is provided with the bilinear form $\langle A, B\rangle = \sum(\langle A(v), B(v)\rangle : v \in V)$. An *isotropic system* is a pair $S = (L, V)$ where V is a finite set and L is a totally isotropic subspace of K^V (i.e. $\langle A, B\rangle = 0$ for every A, $B \in L$) such that $\dim(L) = |V|$.

A vector $A \in K^V$ is said to be *complete* if $A(v) \neq 0$ for every $v \in V$. For $P \subseteq V$ let $AP \in K^V$ be defined by $AP(v) = A(v)$ if $v \in P$ and $AP(v) = 0$ in $v \notin P$. Let $\hat{A} = \{AP : P \subseteq V\}$ and notice that $\hat{A}$ is a subspace of K^V. If A is complete and $\dim(L \cap \hat{A}) = 0$ then A is called an *Eulerian vector* of S. The reader may refer to [3] for a correspondence between 4-regular graphs and isotropic systems where Eulerian vectors correspond to Euler tours.

Two vectors A, $B \in K^V$ are *supplementary* if $0 \neq A(v) \neq B(v) \neq 0$ for every $v \in V$. Let (F, A, B) be a triple with a simple graph F and two supplementary vectors A, $B \in K^V$. Where n is the neighborhood function of F and

$$L = \{An(P) + BP : P \subseteq V\},$$

it is easy to verify that $S = (L, V)$ is an isotropic system (see [3] for details). We call (F, A, B) a *graphic presentation* of S and F a *fundamental graph* of S.

(2.1) *If (F, A, B) is a graphic presentation of an isotropic system S, then A is an Eulerian vector of S. Conversely if A is an Eulerian vector of S, then there exists a graphic presentation (F', A', B') such that $A' = A$, and this graphic presentation is unique.* ∎

(2.2) *Let A be an Eulerian vector of the isotropic system $S = (L, V)$, and let $v \in V$. There exists precisely one Eulerian vector A' satisfying $A'(v) \neq A(v)$ and $A'(w) = A(w)$ for every $w \in V \setminus \{v\}$.* ∎

We use the notation $A * v$ to represent A' of (2.2). For any word $m = v_1 v_2 \ldots v_q$ on V, we let $A * m = (((A * v_1) * v_2) * \ldots) * v_q$.

(2.3) *If A and A' are any two Eulerian vectors of an isotropic system $S = (L, V)$, then there exists a word m on V such that $A' = A * m$.* ∎

(2.4) *Let $P = (F, A, B)$ be a graphic presentation of an isotropic system $S = (L, V)$, and let $v \in V$. The graphic presentation of S induced by the Eulerian vector $A * v$ is $P * v = (F * v, A + Bv, B + An(v))$ (so that $A * v = A + Bv$).* ∎

As a consequence of (2.3) and (2.4), the set of the fundamental graphs of a same isotropic system is a class of local equivalence. Property (2.1) says that any Eulerian vector determines precisely one fundamental graph, but conversely there may be more than one Eulerian vector associated to a same fundamental graph. The following property is a particular case of (5.4) in [4].

(2.5) *For any isotropic system S, there exists an integer k such that any fundamental graph F of S is associated to precisely k Eulerian vectors of S.* ∎

We call the integer k of (2.5) the index of the isotropic system S. It is used to count the number of graphs locally equivalent to a given graph (see [5]).

3. A characterization of local equivalence

If we consider a pair (A_1, B_1) of supplementary vectors of K^V and a vector $A \in K^V$, then we easily verify that there are two uniquely defined subsets $X, Y \subseteq V$ such that $A = A_1X + B_1Y$. We call $A_1X + B_1Y$ the *decomposition* of A over (A_1, B_1).

(3.1) *Let (A_1, B_1) be a pair of supplementary vectors of K^V, and let A_2, $B_2 \in K^V$ be decomposed over (A_1, B_1) as*

$$A_2 = A_1X + B_1Y,$$
$$B_2 = A_1Z + B_1T.$$

For (A_2, B_2) to be a pair of supplementary vectors it is necessary and sufficient that

$$X \cap T + Y \cap Z = V.$$

Proof. For any $P \subseteq V$ we let $\overline{P} = V \setminus P$. For A_2 to be complete it is necessary and sufficient that $X \cup Y = V$, which is equivalent to
(i) $\overline{Y} \subseteq X$.
Similarly B_2 is complete if and only if
(ii) $\overline{Z} \subseteq T$.

Let $C_1 = A_1 + B_1$. For any complete vector A there is a unique decomposition $A = A_1P + B_1Q + C_1R$ with $\{P, Q, R\}$ a partition of V. If we consider another complete vector A' and the decomposition $A' = A_1P' + B_1Q' + C_1R'$ with $\{P', Q', R'\}$ a partition of V, then we have $A(v) \neq A'(v)$ for every $v \in V$ if and only if $P \cap P' = Q \cap Q' = R \cap R' = \emptyset$. Let $A = A_2$ and $A' = B_2$. We have

$$A_2 = A_1[X \cap \overline{Y}] + B_1[Y \cap \overline{X}] + C_1[X \cap Y],$$
$$B_2 = A_1[Z \cap \overline{T}] + B_1[T \cap \overline{Z}] + C_1[Z \cap T],$$

and the above set of characteristic conditions for $A_2(v) \neq B_2(v)$, $v \in V$, becomes
(iii) $X \cap \overline{Y} \cap Z \cap \overline{T} = \emptyset$.
(iv) $Y \cap \overline{X} \cap T \cap \overline{Z} = \emptyset$,
(v) $X \cap Y \cap Z \cap T = \emptyset$.
Taking (i) and (ii) into account, (iii) is equivalent to

(iii') $\overline{Y} \subseteq T$,

and (iv) is equivalent to

(iv') $\overline{Z} \subseteq X$.

The set of the four inclusions (i), (ii), (iii') and (iv') is equivalent to

(v') $\overline{Y} \cup \overline{Z} \subseteq X \cap T$,

when (v) is equivalent to

(v") $\overline{Y} \cup \overline{Z} \supseteq X \cap T$.

Thus the set of the conditions (i)–(v) is equivalent to the equality $\overline{Y} \cup \overline{Z} = X \cap T$, which may be written $X \cap T + Y \cap Z = V$. ∎

For a finite set P we let $|P|_2$ denote the residue class of $|P| \pmod 2$.

(3.2) *Let $S_1 = (L_1, V)$ and $S_2 = (L_2, V)$ be two isotropic systems on the same set V. Let (F_i, A_i, B_i) be a graphic presentation of S_i and let n_i be the neighborhood function of F_i for $i = 1$, 2. Let*

$$A_2 = A_1X + B_1Y,$$
$$B_2 = A_1Z + B_1T.$$

The isotropic systems S_1 and S_2 are equal if and only if

$$|Y \cap n_1(x_1) \cap n_2(x_2)|_2 + |T \cap n_1(x_1) \cap x_2|_2 + |X \cap n_2(x_2) \cap x_1|_2 + |Z \cap x_1 \cap x_2|_2 = 0$$

for every x_1 and x_2 in V.

Proof. The set $E_i = \{A_i n_i(x_i) + B_i x_i : x_i \in V\}$ is a base of the vector-space L_i for $i = 1$, 2. L_1 and L_2 are maximally isotropic subspaces of K^V, and the bilinear form $(A, B) \to \langle A, B\rangle$ defined over K^V is nondegenerate. Thus for $L_1 = L_2$ it is necessary and sufficient that each vector of E_1 is orthogonal to each vector of E_2, which is equivalent to

(i) $\langle A_1n_1(x_1), A_2n_2(x_2)\rangle + \langle A_1n_1(x_1), B_2x_2\rangle + \langle B_1x_1, A_2n_2(x_2)\rangle + \langle B_1x_1, B_2x_2\rangle = 0$ for all x_1 and $x_2 \in V$.

Let us consider in general two subsets P, $Q \subseteq V$. We have

$$\begin{aligned}\langle A_1P, A_2Q\rangle &= \langle A_1P, A_1[X \cap Q] + B_1[Y \cap Q]\rangle \\ &= \langle A_1P, A_1[X \cap Q]\rangle + \langle A_1P, B_1[Y \cap Q]\rangle.\end{aligned}$$

For any R, $S \subseteq V$ we verify that $\langle A_1R, A_1S\rangle = 0$ and $\langle A_1R, B_1S\rangle = |R \cap S|_2$. Thus

(ii) $\langle A_1P, A_2Q\rangle = |Y \cap P \cap Q|_2$,

and similarly

(iii) $\langle A_1P, B_2Q\rangle = |T \cap P \cap Q|_2$,
(iv) $\langle B_1P, A_2Q\rangle = |X \cap P \cap Q|_2$,
(v) $\langle B_1P, B_2Q\rangle = |Z \cap P \cap Q|_2$.

Using (ii)–(v), Equality (i) is equivalent to (4.2.1). ∎

Remark. From now on the permutation (Y, T, X, Z) used in (3.1) and (3.2) will be replaced for convenience by (X, Y, Z, T).

(3.3) *Let F_1 and F_2 be two simple graphs defined over the same vertex-set V, and let n_1 and n_2 be the neighborhood functions of F_1 and F_2 respectively. For F_1 and F_2 to be locally equivalent it is necessary and sufficient that we can find 4 subsets X, Y, Z, $T \subseteq V$ such that*

$$(3.3.1) \qquad |X \cap n_1(x_1) \cap n_2(x_2)|_2 + |Y \cap n_1(x_1) \cap x_2| + |Z \cap n_2(x_2) \cap x_1|_2 + |T \cap x_1 \cap x_2|_2 = 0$$

for every x_1, $x_2 \in V$,

$$(3.3.2) \qquad X \cap T + Y \cap Z = V.$$

In addition if $F_1 = F_2$ then the number k of the solutions (X, Y, Z, T) satisfying (3.3.1) and (3.3.2) is equal to the index of any isotropic system having $F_1 = F_2$ as a fundamental graph. ∎

Proof. Let us consider a pair (A_1, B_1) of supplementary vectors of K^V and the isotropic system S_1 defined by the graphic presentation $(F_1, A_1, B_1,)$. Suppose that F_2 is locally equivalent to F_1. There exists a word m on V such that $F_2 = F_1 * m$. Consider the Eulerian vector $A_2 = A_1 * m$ of S_1 and the graphic presentation of S_1 which is associated to A_2, say (F_2', A_2, B_2). Property (2.4) implies $F_2' = F_2$. Thus (F_1, A_1, B_1) and (F_2, A_2, B_2) are graphic presentations of the same isotropic system S_1. If we define X, Y, Z, $T \subseteq V$ in such a way that

(i) $A_2 = A_1 Z + B_1 X$,
(ii) $B_2 = A_1 T + B_1 Y$,

then (3.1) and (3.2) imply (3.3.1) and (3.3.2).

Conversely if there exist X, Y, Z, T satisfying (3.3.1) and (3.3.2), then we consider A_2 and B_2 defined by (i) and (ii). It follows from (3.1) that (A_2, B_2) is a pair of supplementary vectors. Thus (F_2, A_2, B_2) is a graphic presentation of some isotropic system S_2. It follows from (3.2) that $S_2 = S_1$. Property (2.1) implies that A_2 is an Eulerian vector of S_2. Following (2.3) there exists a word m on V such that $A_2 = A_1 * m$. Then (2.4) implies $F_2 = F_1 * m$. ∎

4. The algorithm

Another way to express (3.3) is to write (3.3.1) and (3.3.2) as a system of equations over $GF(2)$. We consider two simple graphs F_1 and F_2 over the same vertex-set $V = \{1, 2, \ldots, n\}$. For v, w, $i \in V$ we define α_i^{vw}, β_i^{vw}, γ_i^{vw}, $\delta_i^{vw} \in GF(2)$ by

$$\alpha_i^{vw} = 1 \iff iv \in E(F_1) \text{ and } iw \in E(F_2),$$
$$\beta_i^{vw} = 1 \iff iv \in E(F_1) \text{ and } i = w,$$
$$\gamma_i^{vw} = 1 \iff i = v \text{ and } iw \in E(F_2),$$
$$\delta_i^{vw} = 1 \iff i = v = w.$$

Then F_1 is locally equivalent to F_2 if and only if we can solve the following system of equations with $4n$ unknowns X_i, Y_i, Z_i, T_i in $GF(2)$, $i \in V$:

$$(4.1) \qquad \sum_{i=1}^{i=n} (\alpha_i^{vw} X_i + \beta_i^{vw} Y_i + \gamma_i^{vw} Z_i + \delta_i^{vw} T_i) = 0$$

for every v, $w \in V$,

(4.2) $$X_iT_i + Y_iZ_i = 1$$

for every $i \in V$.

The set of the solutions to (4.1) is a subspace $\mathscr{S}$ of $GF(2)^{4n}$. By using a pivoting method, a base B of $\mathscr{S}$ can be computed in $O(n^4)$ time because there are $O(n^2)$ equations in (4.1). Then we can check each vector of $\mathscr{S}$ against Condition (4.2) to find an eventual solution. But the dimension of $\mathscr{S}$ can be equal to $O(n)$, so that the enumeration of $\mathscr{S}$ is nonpolynomial in general. Fortunately we will prove the following result in Section 6:

(4.3) *If the system of equations (4.1)–(4.2) has any solution and* $\dim(\mathscr{S}) > 4$, *then there exists an affine subspace* $\mathscr{A}$ *of* $\mathscr{S}$ *such that* $\dim(\mathscr{S}) - \dim(\mathscr{A}) \leq 2$ *and every* $a \in \mathscr{A}$ *is a solution to (4.1)–(4.2).* ∎

Then we use the following lemma.

(4.4) *For every base* B *of a vector space* $\mathscr{S}$ *over* $GF(2)$ *and every affine subspace* $\mathscr{A}$ *of* $\mathscr{S}$ *such that* $\dim(\mathscr{S}) - \dim\mathscr{A}) \leq q$, *there exists a vector* $a \in \mathscr{A}$ *which is the sum of* $\leq q$ *vectors of* B.

Proof. The set $W = \{x - y : x, y \in \mathscr{A}\}$ is a subspace of $\mathscr{S}$. Consider the canonical projection $p : \mathscr{S} \to \mathscr{S}/W$. The image $p(B)$ is a generating set of $\mathscr{S}/W$. So we can find a base $\{p(b_1), p(b_2), \ldots, p(b_k)\}$ of $\mathscr{S}/W$ with $b_1, b_2, \ldots, b_k \in B$. We have $k = \dim(\mathscr{S}/W) = \dim(\mathscr{S}) - \dim(\mathscr{A}) \leq q$. Since $\mathscr{A} \in \mathscr{S}/W$, we can find $I \subseteq \{1, 2, \ldots, k\}$ such that $\mathscr{A} = \sum(p(b_i) : i \in I)$. This implies the existence of $a \in A$ such that $a = \sum(b_i : i \in I)$. ∎

Thus to find an eventual solution σ to (4.1)–(4.2) we enumerate either the ≤ 16 elements of $GF(2)^{4n}$ such belong to $\mathscr{S}$ if $\dim(\mathscr{S}) \leq 4$ or the $O(n^2)$ elements of $GF(2)^{4n}$ which are decomposable as a sum of ≤ 2 vectors of B, and for each of these elements we check in $O(n)$ time whether Condition (4.2) is satisfied. The overall complexity does not exceed $O(n^3)$, which is lower than $O(n^4)$, the complexity for computing B.

To find a word m such that $F_2 = F_1 * m$ we adapt a technique of Fon-Der-Flaass [7]. Choose a pair (A_1, B_1) of supplementary vectors in K^V and determine the pair of supplementary vectors (A_2, B_2) by means of the solution σ and the decomposition of A_2 and B_2 over (A_1, B_1) given in (3.1). Thus we get two graphic presentations $P_1 = (F_1, A_1, B_1)$ and $P_2 = (F_2, A_2, B_2)$ of a same isotropic system S. We define the *divergence* from A_2 to A_1 as the function $d : V \to \{0, 1, 2\}$ satisfying the following relations for every $x \in V$:

$$d(x) = 0 \text{ if } A_2(x) = A_1(x),$$
$$d(x) = 1 \text{ if } A_2(x) = A_1(x) + B_1(x),$$
$$d(x) = 2 \text{ if } A_2(x) = B_1(x).$$

Notice that $d(x) = 1$ also means $A_2 * x(x) = A_1(x)$ following (2.4). Let $v \in V$ and consider the graphic presentation $P_2' = P_2 * v = (F_2', A_2', B_2')$. The divergence d' from

A'_2 to A_1 can be computed by the following formulas easily derived from (2.4):

$$d'(v) = d(v) \text{ if } d(v) = 2 \text{ otherwise } d'(v) = 1 - d(v),$$
$$d'(x) = d(x) \text{ if } d(x) = 0 \text{ otherwise } d'(x) = 3 - d(x),\ x \in n_2(v),$$
$$d'(x) = d(x)\ x \neq v \text{ and } x \notin n_2(v).$$

Fon-Der-Flaass' algorithm [7] is the following one:

if there exists v such that $d(v) = 1$ then replace F_2 by $F_2 * v$ (and so $d(v)$ becomes equal to 0);

if there exists an edge vw such that $d(v) = d(w) = 2$ then replace F_2 by $F_2 * vwv$ (and so $d(v)$ and $d(w)$ become equal to 0).

The algorithm stops when there is no longer any value $d(v) = 1$ and the subset $I = \{x : d(x) = 2\}$ is independent. Since F_2 is locally equivalent to F_1, Fon-Der-Flaass' results imply that $I = \emptyset$ and the final graph is equal to F_1. The algorithm is greedy, and so it takes $O(n)$ steps. Each step requires a local complementation of complexity which is no more than $O(n^2)$. The overall complexity does not exceed $O(n^3)$.

5. The linear bijection β

Our main task is now to prove (4.3). For that we use again the compact notation defined in Section 3. We consider $\mathcal{P}(V)^4 = \{(X, Y, Z, T) : X,\ Y,\ Z,\ T \subseteq V\}$ with its canonical structure of vector space over $GF(2)$. The bijection between $\mathcal{P}(V)^4$ and $GF(2)^{4n}$, considered in Section 4, is the natural one. The mapping

$$((X, Y, Z, T), (X', Y', Z', T')) \to \langle (X, Y, Z, T), (X', Y', Z', T') \rangle =$$
$$|X \cap X'|_2 + |Y \cap Y'|_2 + |Z \cap Z'|_2 + |T \cap T'|_2$$

is a symmetric bilinear form. For any subspace N of $\mathcal{P}(V)^4$ we let $N^\perp = \{\Phi : \Phi \in \mathcal{P}(V)^4,\ \langle \phi, \psi \rangle = 0,\ \psi \in N\}$.

For $x_1,\ x_2 \in V$, we let

$$\lambda(x_1, x_2) = (n_1(x_1) \cap n_2(x_2), n_1(x_1) \cap x_2, n_2(x_2) \cap x_1, x_1 \cap x_2)$$

and denote by $\lambda(F_1, F_2)$ the subspace of $\mathcal{P}(V)^4$ generated by $\{\lambda(x_1, x_2) : x_1, x_2 \in V\}$. Thus $\lambda(F_1, F_2)$ is the set of the solutions to (3.3.1), and it corresponds to the set $\mathcal{S}$ considered in Section 4. We denote by $\sigma(F_1, F_2)$ the subset of the solutions to (3.3.1) and (3.3.2) (which corresponds to the set of the solutions to (4.1) and (4.2)).

For $\phi = (X, Y, Z, T) \in \mathcal{P}(V)^4$ we let

$$\phi^{[1]} = (Z, T, X, Y),$$
$$\phi^{[2]} = (Y, X, T, Z).$$

For $\alpha = (A, B, C, D) \in \mathcal{P}(V)^4$, $i \in \{1, 2\}$ and N a subspace of $\mathcal{P}(V)^4$ let

$$\phi \cap \alpha = (X \cap A, Y \cap B, Z \cap C, T \cap D),$$
$$\phi \underset{i}{*} \alpha = \phi + \phi^{[i]} \cap \alpha,$$
$$N \underset{i}{*} \alpha = \{\phi \underset{i}{*} \alpha : \phi \in N\}.$$

When α is fixed the map $\phi \to \phi \underset{i}{*} \alpha$ is linear, so that $N \underset{i}{*} \alpha$ is a subspace of $\mathcal{P}(V)^4$.

(5.1) *Let $i \in \{1,2\}$ and $\alpha \in \mathcal{P}(V)^4$ be such that $\alpha \cap \alpha^{[i]} = 0$. For ϕ, $\psi \in \mathcal{P}(V)^4$ and a subspace N of $\mathcal{P}(V)^4$, the following properties hold:*

(i) $\phi \underset{i}{*} \alpha \underset{i}{*} \alpha = \phi$;

(ii) $\phi \to \phi \underset{i}{*} \alpha$ *is bijective;*

(iii) $\langle \phi \underset{i}{*} \alpha, \psi \underset{i}{*} \alpha^{[i]} \rangle = \langle \phi, \psi \rangle$;

(iv) $(N \underset{i}{*} \alpha)^{\perp} = N^{\perp} \underset{i}{*} \alpha^{[i]}$.

Proof. To verify (i) and (iii) is easy. Then (i) implies (ii) and (iii) implies (iv). ∎

(5.2) *Let F_1 and F_2 be two simple graphs on the same vertex-set V with neighborhood functions n_1 and n_2 respectively, and let $v \in V$. Then*

(i) $\lambda(F_1 * v, F_2) = \lambda(F_1, F_2) \underset{1}{*} (n_1(v), n_1(v), v, v)$;

(ii) $\lambda(F_1, F_2 * v) = \lambda(F_1, F_2) \underset{2}{*} (n_2(v), v, n_2(v), v)$;

(iii) $\lambda(F_1 * v, F_2)^{\perp} = \lambda(F_1, F_2)^{\perp} \underset{1}{*} (v, v, n_1(v), n_1(v))$;

(iv) $\lambda(F_1, F_2 * v)^{\perp} = \lambda(F_1, F_2)^{\perp} \underset{2}{*} (v, n_2(v), v, n_2(v))$.

Proof. We just prove (i) since (ii) is similar, and (iii) and (iv) then follow from (5.1). Let $F_1' = F_1 * v$ and let n_1' be the neighborhood function of F_1'. The subspace $\lambda(F_1', F_2)$ is generated by $(\lambda'(x_1, x_2) : x_1, x_2 \in V)$ where

$$\lambda'(x_1, x_2) = (n_1'(x_1) \cap n_2(x_2), n_1'(x_1) \cap x_2, x_1 \cap n_2(x_2), x_1 \cap x_2).$$

We easily verify that

$$n_1'(x_1) = n_1(x_1) + e_{x_1 v} n_1(v) + n_1(v) \cap x_1$$

for every $x_1 \in V$, $e_{x_1 v} = 1$ if $x_1 v$ is an edge of F_1 and $e_{x_1 v} = 0$ otherwise. Thus

$$\lambda'(x_1, x_2) = \lambda(x_1, x_2) + e_{x_1 v}(n_1(v) \cap n_2(x_2), n_1(v) \cap x_2, 0, 0) + (x_1 \cap n_1(v) \cap n_2(x_2), x_1 \cap n_1(v) \cap x_2, 0, 0).$$

Another generating family of $\lambda(F_1', F_2)$ is $(\lambda''(x_1, x_2) = \lambda'(x_1, x_2) + e_{x_1 v}\lambda'(v, x_2) : x_1, x_2 \in V)$. We have

$$\begin{aligned}\lambda''(x_1, x_2) &= \lambda(x_1, x_2) + e_{x_1 v}(0, 0, v \cap n_2(x_2), v \cap x_2) + \\ &\quad + (x_1 \cap n_1(v) \cap n_2(x_2), x_1 \cap n_1(v) \cap x_2, 0, 0) \\ &= \lambda(x_1, x_2) + (0, 0, v \cap n_1(x_1) \cap n_2(x_2), v \cap n_1(x_1) \cap x_2) \\ &\quad + (x_1 \cap n_1(v) \cap n_2(x_2), x_1 \cap n_1(v) \cap x_2, 0, 0) \\ &= \lambda(x_1, x_2) \underset{1}{*} (n_1(v), n_1(v), v, v),\end{aligned}$$

which proves the property since $(\lambda(x_1, x_2) : x_1, x_2 \in V)$ is a generating family of $\lambda(F_1, F_2)$. ∎

Definition. Let (F_1, F_2) and (F_1', F_2') be two pairs of simple graphs defined over a same vertex-set V, and suppose that $F_1' = F_1 * m_1$ and $F_2' = F_2 * m_2$, where m_1 and m_2 are words with letters in V. We define a linear bijection $\beta = \beta(F_1, F_1', F_2, F_2')$ from $\mathcal{P}(V)^4$ into $\mathcal{P}(V)^4$ as follows:

(i) if $m_1 = v$, $v \in V$, and m_2 is the empty word, then

$$\beta : \phi \to \phi \underset{1}{*} (v, v, n_1(v), n_1(v));$$

(ii) if $m_2 = v$, $v \in V$, and m_1 is the empty word, then

$$\beta : \phi \to \phi \underset{2}{*} (v, n_2(v), v, n_2(v));$$

(iii) in the other cases β is defined by composition from Cases (i) and (ii).

Then (5.2) implies

(5.3) $\lambda(F_1', F_2')^{\perp} = \beta(\lambda(F_1, F_2)^{\perp})$.

(5.4) *With the above notation let (X', Y', Z', T') be the image of some $(X, Y, Z, T) \in \mathcal{P}(V)^4$ by the linear bijection $\beta(F_1, F_1', F_2, F_2')$. Then*

$$X' \cap T' + Y' \cap Z' = X \cap T + Y \cap Z.$$

Proof. Let $F_2' = F_2$ and $F_1' = F_1 * v$, $v \in V$. We have

$$(X', Y', Z', T') = (X, Y, Z, T) + (Z, T, X, Y) \cap (v, v, n_1(v), n_1(v)),$$

which implies

$$\begin{aligned} X' \cap T' &= (X + Z \cap v) \cap (T + Y \cap n_1(v)) \\ &= (X \cap T + X \cap Y \cap n_1(v) + Z \cap T \cap v, \\ Y' \cap Z' &= (Y + T \cap v) \cap (Z + X \cap n_1(v)) \\ &= Y \cap Z + X \cap Y \cap n_1(v) + Z \cap T \cap v, \end{aligned}$$

which in turn implies the equality of the statement. The verification is similar for $F_1 = F_1'$ and $F_2 = F_2' * v$. It is obtained by composition for general $\beta(F_1, F_1', F_2, F_2')$. ∎

Following Condition (3.3.2), an element $(X, Y, Z, T) \in \lambda(F_1, F_2)^{\perp}$ also belongs to $\sigma(F_1, F_2)$ if and only if $X \cap T + Y \cap Z = V$. Therefore the preceding property implies

(5.5) $\sigma(F_1', F_2') = \beta(\sigma(F_1, F_2))$.

6. Internal solutions (Bineighbourhood Space)

We now consider a simple graph F over the vertex-set V and we are interested in $\sigma(F,F)$, the set of the *internal solutions* w.r.t. F. We suppose that F is connected, which is not a restriction because local complementations preserve connected components. To simplify the notation we let $\Lambda(F) = \lambda(F,F)^{\perp}$. Where n is the neighborhood function of F, we let $\nu(xy) = n(x) \cap n(y)$ for every nonordered pair of distinct vertices x and y. We denote by $\overline{F}$ the complementary of the simple graph F. Any $P \in \mathcal{P}(V)$ will be identified to its characteristic function with values in $GF(2)$, so that for every $x \in P$ we have $P(x) = 1$ if $x \in P$, $P(x) = 0$ otherwise. For P, $Q \in \mathcal{P}(V)$ we let $\langle P,Q\rangle = |P \cap Q|_2$, and for any subspace N of $\mathcal{P}(V)$ we let $N^{\perp} = \{P \in \mathcal{P}(V) : \langle P,Q\rangle = 0 \text{ for every } Q \in N\}$. Following (3.3.1) an element (X,Y,Z,T) of $\mathcal{P}(V)^4$ belongs to $\Lambda(F)$ if and only if it satisfies the following conditions

(6.1) $$\langle X,\nu(xy)\rangle = Z(x) + Y(y), \quad xy \text{ is an edge of } F;$$

(6.2) $$\langle X,\nu(xy)\rangle = 0, \quad xy \text{ is an edge of } \overline{F};$$

(6.3) $$\langle X,n(x)\rangle = T(x), \quad x \text{ is a vertex of } F.$$

We easily verify that (6.3) is equivalent to

(6.3)′ $$T = n(X).$$

(6.4) *Every element $(X,Y,Z,T) \in \Lambda(F)$ either satisfies $Z = Y$ or $Z = \overline{Y}$.*

Proof. Following (6.1) we have $\langle X,\nu(xy)\rangle = Z(x) + Y(y)$ and $\langle X,\nu(yx)\rangle = Z(y) + Y(x)$ for every edge xy, which implies $Z(y) - Y(y) = Z(x) - Y(x)$. Since F is connected $Z(x) - Y(x)$ will be equal to a constant k. If $k = 0$ we have $Z = Y$, otherwise $Z = \overline{Y}$. ∎

We let

$$\Lambda_0(F) = \{(X,Y,Z,T) \in \Lambda(F) : Z = Y\},$$
$$\Lambda_1(F) = \{(X,Y,Z,T) \in \Lambda(F) : Z = \overline{Y}\}.$$

We notice that $\Lambda_0(F)$ is a sunspace of $\Lambda(F)$. The preceding property is equivalent to

(6.4)′ $$\Lambda(F) = \Lambda_0(F) \cup \Lambda_1(F).$$

For any cycle C of F we let $\nu(C) = \sum(\nu(xy) : xy \in C\}$ and we denote by $\nu(F)$ the subspace of $\mathcal{P}(V)$ generated by $\{\nu(C) : C \text{ is a cycle of } F\} \cup \{\nu(xy) : xy \text{ is an edge of } \overline{F}\}$. We call $\nu(F)$ the *bineighbourhood space* of F, and we study this space with some detail in [5].

Lemma (6.5) *Let F be a simple graph over the vertex-set V. A subset $X \subseteq V$ belongs to $\nu(F)^{\perp}$ if and only if there exists $(X,Y,Z,T) \in \Lambda_0(F)$ with this given X. Moreover $(X,Y,Z,T) \in \Lambda_0(F)$ implies $(X,\overline{Y},\overline{Y},T) \in \Lambda_0(F)$, and there is no other element of $\Lambda_0(F)$ with this given X.*

Proof. Let us consider some $(X,Y,Y,T) \in \Lambda_0(F)$. For (6.1) to be satisfied it is necessary that

(a) $\sum(\langle X, \nu(xy)\rangle : xy \in C) = 0$, C is a cycle of F,

which also may be written $\langle X, \nu(C)\rangle = 0$. Thus for (6.1) and (6.2) to be satisfied, it is necessary that $X \in \nu(F)^{\perp}$. Conversely let $X \in \nu(F)^{\perp}$ and choose any value $Y(v_0)$ at some vertex v_0 of F. It follows from (a) that we can consistently define a subset $Y \subseteq V$ by letting $Y(v) = Y(v_0) + \sum(\langle X, \nu(xy)\rangle : xy \in P)$ for each vertex v and P any path from v_0 to v. Then $Z = Y$ satisfies (6.1). Equality (6.2) is satisfied because $X \in \nu(F)^{\perp}$, and(6.3)' will hold by choosing the appropriate T. Therefore we can actually construct $(X, Y, Y, T) \in \Lambda_0(F)$ with the given X. Finally we notice that Y is uniquely determined when the value $Y(v_0)$ is chosen, and the two possible solutions corresponding to the two possible values of $Y(v_0)$ are complementary subsets. ∎

For every $X \in \mathcal{P}(V)$ let

$$\Lambda(X, F) = (X, X \cap n(X), X \cap n(X), n(X)),$$
$$\overline{\Lambda}(X, F) = (X, \overline{X \cap n(X)}, \overline{X \cap n(X)}, n(X)),$$

(6.6) *For any simple graph F and any $X \in \nu(F)^{\perp}$, $\Lambda(X, F)$ belongs to $\Lambda_0(F)$.*

Proof. $\Lambda(X, F)$ obviously satisfies (6.3)'. Since $X \in \nu(F)^{\perp}$, it also satisfies (6.2). Thus it remains to prove that (6.1) is satisfied for $Z = Y = X \cap n(X)$ when $X \in \nu(F)^{\perp}$. The proof proceeds through three steps.

Claim (6.6.1) *If X is an independent subset of F, then $\Lambda(X, F) \in \Lambda_0(F)$.*

Proof. In this case we have $Z = Y = \emptyset$, so that (6.1) will hold if we prove

(b) $\langle X, \nu(xy)\rangle = 0$, xy is an edge of F.

Case 1. Either $x \in X$ or $y \in X$. Equality (b) holds because no $z \in X$ is joined to both x and y.

Case 2. $x \notin X$ and $y \notin X$. If no $z \in X$ is joined to both x and y, then (b) obviously holds; otherwise choose such a z and consider the cycle $C = (x, y, z)$. We have $\langle X, \nu(C)\rangle = 0$ because $X \in \nu(F)^{\perp}$. Therefore

$$\langle X, \nu(xz)\rangle + \langle X, \nu(zy)\rangle + \langle X, \nu(xy)\rangle = 0.$$

It follows from Case 1 that the two first terms vanish in the above sum, so that the third one also vanishes. ∎

Claim (6.6.2) *Let $F' = F * v$ for some vertex v, and let $\beta = \beta(F, F', F, F')$. We have*

$$\beta(\Lambda(X, F)) = \Lambda(X + n(X) \cap v, F').$$

Proof. It is essentially a computation. We first transform $\psi_0 = \Lambda(X, F)$ by $\beta(F, F', F, F)$, and we transform the image ψ_1 by $\beta(F', F', F, F')$ to obtain $\psi_2 = \beta(\Lambda(X, F))$. To simplify the notation we let $Y = X \cap n(X)$ and $T = n(X)$. We have

$$\begin{aligned}\psi_1 &= \psi_0 + \psi_0^{[1]} \cap (v, v, n(v), n(v)) \\ &= (X, Y, Y, T) + (Y, T, X, Y) \cap (v, v, n(v), n(v)).\end{aligned}$$

To compute ψ_2 from ψ_1 we have to use the neighborhood function n' of F' but we notice that $n'(v) = n(v)$. We also use the property $n(v) \cap v = \emptyset$.

$$\begin{aligned}\psi_2 &= \psi_1 + \psi_1^{[2]} \cap (v, n(v), v, n(v)) \\ &= (X, Y, Y, T) + (Y, T, X, Y) \cap (v, v, n(v), n(v)) + \\ &\quad (Y, X, T, Y) \cap (v, n(v), v, n(v)) + \\ &\quad (T, Y, Y, X) \cap (v, v, n(v), n(v)) \cap (v, n(v), v, n(v)) \\ &= (X + T \cap v, Y + T \cap v + X \cap n(v), \\ &\quad Y + T \cap v + X \cap n(v), T + X \cap n(v)).\end{aligned}$$

The neighborhood function n' is related with n by

$$n'(x) = n(x) + \langle x, n(v) \rangle n(v) + x \cap n(v), \quad x \in V,$$

so that

$$n'(X') = n(X') + \langle X', n(v) \rangle n(v) + X' \cap n(v), \quad X' \subseteq V.$$

Applying the preceding formula to $X' = X + T \cap v$, the reader will verify that $\Lambda(X', F') = \psi_2$. ∎

We verify that the linear mapping $X \to X + n(X) \cap v$ is bijective. Thus the preceding property implies that $\beta = \beta(F, F', F, F')$ maps $\Lambda_0(F)$ onto $\Lambda_0(F')$ when $F' = F * v$. This also holds for any F' locally equivalent to F by composition.

Claim (6.6.3) *Let F be a simple graph over the vertex-set V. For every $X \subseteq V$ there exists F' locally equivalent to F such that if we let $\beta = \beta(F, F', F, F')$ and we define X' by $\Lambda(X', F') = \beta(\Lambda(X, F))$, then X' is an independent subset of F'.*

Proof. To prove the property we may replace the pair (X, F) by any pair (X', F') with F' locally equivalent to F and $\Lambda(X', F') = \beta(\Lambda(X, F))$. We choose (X, F) so that $|X|$ is minimal. Then we show that X is independent in F, which will prove the claim with $X' = X$ and $F' = F$.

There cannot exist a vertex v of odd degree in the induced subgraph $F[X]$. On the contrary we should have $v \in n(X)$. Taking $F' = F * v$, it follows from (6.6.2) that $X' = X + n(X) \cap v = X \setminus v$, so that $|X'| < |X|$, a contradiction with the choice of X.

Thus every vertex of $F[X]$ has even degree, so that $n(X) \cap X = \emptyset$. Suppose that some edge vw does exist in $F[X]$, and replace F by $F' = F * v$. We have $X' = X + n(X) \cap v = X$ because $n(X) \cap X = \emptyset$. But after locally complementing F at v, the vertex w of even degree in $F[X]$ becomes of odd degree in $F'[X']$, so that we can repeat the above argument with (X', F') replacing (X, F), again a contradiction with the minimality of $|X| = |X'|$. ∎

To prove the proposition we apply (6.6.3) to find F' locally equivalent to F and X' independent in F' such that $\Lambda(X', F') = \beta(\Lambda(X, F))$ with $\beta = \beta(F, F', F, F')$. It follows from (6.6.1) that $\Lambda(X', F') \in \Lambda_0(F')$. But $\beta^{-1} = \beta(F', F, F', F)$ maps $\Lambda_0(F')$ onto $\Lambda_0(F)$, so that $\Lambda(X, F) \in \Lambda_0(F)$. ∎

It follows from (6.5) and (6.6) that

(6.7) $\Lambda_0(F) = \{\Lambda(X, F) : X \in \nu(F)^{\perp}\} \cup \{\overline{\Lambda}(X, F) : X \in \nu(F)^{\perp}\}$.

We notice that $\Lambda(X, F)$ does not satisfy Condition (3.3.2) when $\overline{\Lambda}(X, F)$ does. We call $\overline{\Lambda}(X, F)$, $X \in \nu(F)^{\perp}$, a *regular solution* to (3.3.1)–(3.3.2).

(6.8) *If* $\dim(\nu(F)^{\perp}) > 2$, *the set of the regular solutions which belong to* $\Lambda_0(F)$ *is an affine subspace of codimension 1 (hyperplane) in* $\Lambda_0(F)$.

To prove (6.8) we use the following lemma of D. F. Fon-Der-Flaass [8].

Lemma (6.8.1) *Let* $\psi(X'X'') = X' \cap n(X'') + X'' \cap n(X')$ *be defined for* X', $X'' \in \nu(F)^{\perp}$, *and suppose that* $\psi(X', X'') \in \{\emptyset, V\}$ *always holds. Then either* $\dim(\nu(F)^{\perp}) = 2$ *or* $\psi(X', X'') = 0$ *always holds.*

Proof. Consider any pair of vectors X', $X'' \in \nu(F)^{\perp}$ such that $\psi(X', X'') = V$.

We claim that any $Y \in \nu(F)^{\perp}$ satisfying $\psi(X', Y) = \emptyset$ is such that $Y \subseteq X'$. Since ψ is bilinear we have $\psi(X', X'' + Y) = V$. We notice that whenever two elements Z, $T \in \nu(F)^{\perp}$ satisfy $\psi(Z, T) = V$, they also satisfy $Y \cup Z = V$. Thus $X' \cup X'' = X' \cup (X'' + Y) = V$, which implies $Y \subseteq X'$.

Let us consider $X = X' + X''$ and any $Y \in \nu(F)^{\perp}$. Since ψ is bilinear, either the three values $\psi(X, Y)$, $\psi(X', Y)$, $\psi(X'', Y)$ are null or two are equal to V when the third one is null. We notice that $\psi(X, X') = \psi(X', X') + \psi(X'', X') = V$. Therefore in the first case we have $Y \subseteq X' \cap X'' \cap X = \emptyset$. Thus if $Y \neq \emptyset$, the second case must occur. We may suppose w.l.o.g. that $\psi(X, Y) = \emptyset$ when $\psi(X', Y) = \psi(X'', Y) = V$. Then we have $Y \subseteq X$ because $\psi(X, X') = V$ and $X \subseteq Y$ because $\psi(Y, X') = V$, so that $Y = X$. It follows that $\nu(F)^{\perp}$ is generated by X' and X'', which proves the lemma. ∎

Proof of (6.8). Since $\Lambda_0(F)$ is a vector space, Equality (6.7) implies that for X', $X'' \in \nu(F)^{\perp}$ we have either

(i) $\Lambda(X'F) + \Lambda(X'', F) = \Lambda(X' + X'', F)$

or

(ii) $\Lambda(X'F) + \Lambda(X'', F) = \overline{\Lambda}(X' + X'', F)$.

Let us consider the mapping ψ defined in the lemma. We verify that Case (i) occurs if $\psi(X', X'') = 0$ when Case (ii) occurs if $\psi(X', X'') = V$. Following the lemma, only Case (i) does occur since we suppose $\dim(\nu(F)^{\perp}) > 2$. Therefore the set $A = \{\Lambda(X, F) : X \in \nu(F)^{\perp}\}$ is a subspace of codimension 1 of $\Lambda_0(F)$. The set $\overline{A} = \{\overline{\Lambda}(X, F) : X \in \nu(F)^{\perp}\}$ is an hyperplane parallel to A, and each element in $\overline{A}$ is a regular solution. ∎

(6.9) *Let* F_1 *and* F_2 *be two simple graphs over the same vertex-set* V. *If* $\sigma(F_1, F_2)$ *is nonempty and* $\dim(\lambda(F_1, F_2)^{\perp}) > 4$, *then* $\sigma(F_1, F_2)$ *includes an affine subspace of codimension* ≤ 2 *in* $\lambda(F_1, F_2)^{\perp}$.

Proof. If $\sigma(F_1, F_2)$ is nonempty then F_1 and F_2 are locally equivalent, so that we can consider some F locally equivalent to F_1 and F_2. Let ϱ be the set of the regular solutions which belong to $\Lambda_0(F)$. We have $\varrho \subseteq \sigma(F, F)$. We have $\dim(\Lambda(F)) = \dim(\lambda(F, F)^{\perp}) = \dim(\lambda(F_1, F_2)^{\perp}) > 4$, which implies $\dim(\nu(F)^{\perp}) > 2$ because (6.7) holds and $\Lambda_0(F)$ is a subspace of codimension ≤ 1 in $\Lambda(F)$. Following

(6.8) ϱ is an hyperplane in $\Lambda_0(F)$. Since $\Lambda_0(F)$ is a subspace of codimension ≤ 1 in $\Lambda(F) = \lambda(F,F)^{\perp}$, ϱ is an affine subspace of codimension ≤ 2 in $\lambda(F,F)^{\perp}$. The linear bijection $\beta(F,F_1,F,F_2)$ maps ϱ into an affine subspace of codimension ≤ 2 in $\lambda(F_1,F_2)^{\perp}$. ∎

Proof of (4.3). It is a rephrasing of (6.9) where $\lambda(F_1,F_2)^{\perp}$ stands for $\mathscr{S}$ and $\sigma(F_1,F_2)$ stands for the set of the solutions to (4.1)–(4.2). ∎

7. Singular solutions

We still use the notation of Section 6. We call an element of $\sigma(F,F) \cap \Lambda_1(F)$ a *singular solution*.

(7.1) *The subset $\Lambda_1(F)$ is nonempty if and only if there exists $X \subseteq V$ such that:*
(i) $\langle X, \nu(C)\rangle = |C|_2$ for every cycle C of F,
(ii) $\langle X, \nu(xy)\rangle = 0$ for every edge xy of $\overline{F}$.

Proof. For $(X,Y,Z,T) \in \Lambda_1(F)$, (6.1) becomes $\langle X, \nu(xy)\rangle = Y(x) + Y(y) + 1$, xy is an edge of F. Then we prove as in (6.5) that (i) must hold. The converse also is proved similarly. ∎

(7.2) *There exist singular solutions if and only if F satisfies the following conditions:*
(i) every vertex of F has an odd degree;
(ii) $|\nu(xy)|_2 = 0$, xy is an edge of $\overline{F}$;
(iii) $|\nu(C)|_2 = |C|_2$, C is a cycle of F.
Then there exist precisely two singular solutions $(V,Y,\overline{Y},V)$ and $(V,\overline{Y},Y,V)$ with Y satisfying

$$|\nu(xy)|_2 + 1 = Y(x) + Y(y).$$ ∎

Proof. For (X,Y,Z,T) to satisfy (3.3.2) when $Z = \overline{Y}$, we must have $X = T = V$, which implies $n(X) = V$ by (6.3)'. This equality amounts to say that F has odd degrees. The other conditions are those of (7.1) with $X = V$. The remaining of the statement is proved like (6.5). ∎

The class of the graphs satisfying the conditions of the preceding statement is called *Class α*. It has been verified directly by L. Allys [1] that Class α is invariant by local complementation. We notice that Condition (iii) of the preceding statement always holds for a bipartite graph. For example if we consider the complete bipartite graph $K_{2m,2m}$, and we delete a perfect matching, then we easily verify that the resulting bipartite graph is in Class α. Let us now characterize the bipartite graphs in Class α.

A *binary matroid* is a pair $M = (N, V)$ with a finite set V and a subspace N of $\mathscr{P}(V)$. The *dual* of M is $M^* = (N^{\perp}, V)$, and we say that M is *autodual* if $M = M^*$ (or equivalently $N = N^{\perp}$). A *base* of M is a subset $X \subseteq V$ such that $n \in N$ and $n \subseteq X$ imply $n = \emptyset$. For any $y \in Y = V \setminus X$ there exists precisely one nonnull $n_y \in N$ such that $n_y \subseteq X \cup \{y\}$. The bipartite graph F, defined on V, whose edges are the pairs xy, $x \in X$, $y \in Y$, $x \in n_y$, is called a *fundamental graph* of M. We notice that F is also a fundamental graph of M^*.

(7.3) *A bipartite graph F is in Class α if and only if it is a fundamental graph of an autodual binary matroid.*

Proof. Let F be defined on the color classes X and Y. For $y \in Y$ let $n_y = \{y\} \cup \{x : xy$ is an edge of $F\}$. If N is the subspace of $\mathcal{P}(V)$ generated by $(n_y : y \in Y)$, then we verify that X is a base of the binary matroid $M = (N, V)$, so that F is a fundamental graph of M. Condition (i) of (7.2) amounts to $\langle n_y, n_y \rangle = 0$ for every $y \in Y$, and Condition (ii) amounts to $\langle n_y, n_z \rangle = 0$ for every y, $z \in Y$. Therefore any two vectors in $(n_y : y \in Y)$ are orthogonal, and since this family is a base of N (in the sense of linear algebra) this implies $N \subseteq N^{\perp}$. Exchanging the roles of X and T, we also obtain $N^{\perp} \subseteq N$. Therefore $N = N^{\perp}$ if F is in Class α. The converse is proved similarly. ∎

Class α also contains graphs which are not locally equivalent to bipartite graphs, for example the 5-wheel.

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